電機系較重要的微積分應用

一般微積分$y(x)$中的自變數x對應電路學的是t(時間變數)，微積分因變數y對應電路學的是電壓v或電流i，而其波形函數則是阻尼弦波函數，阻尼弦波函數為 $f(t) = M \times e^{-at}\cos(bt) + N \times e^{-at}\sin(bt)$

電機系常用以下阻尼弦波函數函數$f(t)$

1. $M \times e^{-at}\cos(bt)$
2. $N \times e^{-at}\sin(bt)$
3. $M \times \cos(bt)$　交流電，相當於 1.中 a=0
4. $N \times \sin(bt)$　交流電，相當於 2.中 a=0
5. $M \times e^{-at}$　相當於 1.中 b=0
6. M　直流電，相當於 1.中 a=0 且 b=0

以下範例首先使用電機系常用的電感L進行說明，其電壓v與電流i有以下關係：

$$v(t) = L \times \frac{di(t)}{dt}$$

必須使用微分與定積分來求解，共有四組對應關係(範例 1 對應範例 2；範例 3 對應範例 4；範例 5 對應範例 6；範例 7 對應範例 8)。另外，電機系常用的電容C其電壓v與電流i有以下關係：

$$i(t) = C \times \frac{dv(t)}{dt}$$

求解方式也很類似電感的八個範例，有興趣的同學請自行練習。

範例 1

電機系常用的電感L其電壓v與電流i有以下關係：

$$v(t) = L \times \frac{di(t)}{dt}$$

已知$L = 2(H)$; $i(t) = 4\cos(2t)$; 求$v(t) =?$

解:

$$v(t) = L \times \frac{di(t)}{dt} = 2 \times \frac{d4\cos(2t)}{dt} = 2 \times 4\frac{d\cos(2t)}{dt} = -16\sin{(2t)}$$

範例 2

電機系常用的電感L其電壓v與電流i有以下關係：

$$v(t) = L \times \frac{di(t)}{dt}$$

已知$L = 2(H)$; $v(t) = -16\sin(2t)$; $i(0) = 4$ 求$i(t) =?$

解:

$$v(t) = L \times \frac{di(t)}{dt}$$

$$-16\sin(2t) = 2 \times \frac{di(t)}{dt}$$ 等式兩邊同時除以 2

$$-8\sin(2t) = \frac{di(t)}{dt}$$

$$-8\sin(2t)dt = di(t)$$ 類似交叉相乘

$$\int_0^t -8\sin(2\tau)d\tau = \int_0^t di(\tau)$$

$$-8\int_0^t \sin(2\tau)d\tau = \int_0^t di(\tau)$$

$$-8\int_0^t \sin(2\tau)d(2\tau) \times \frac{1}{2} = \int_0^t di(\tau)$$

$$-4\int_0^t \sin(2\tau)d(2\tau) = \int_0^t di(\tau)$$

$$-4 \times (-\cos(2\tau))\Big|_0^t = i(\tau)\Big|_0^t$$

$$4\cos(2t) - 4 = i(t) - i(0) = i(t) - 4$$

$$i(t) = 4\cos(2t)$$

範例 3

電機系常用的電感L其電壓v與電流i有以下關係：

$$v(t) = L \times \frac{di(t)}{dt}$$

已知$L = 2(H)$; $i(t) = 4\sin(2t)$；求$v(t) =?$

解:

$$v(t) = L \times \frac{di(t)}{dt} = 2 \times \frac{d4\sin(2t)}{dt} = 2 \times 4\frac{d\sin(2t)}{dt} = 16cos(2t)$$

<u>範例 4</u>

電機系常用的電感L其電壓v與電流i有以下關係：

$$v(t) = L \times \frac{di(t)}{dt}$$

已知$L = 2(H)$; $v(t) = 16\cos(2t)\,;i(0) = 0$ 求$i(t) =?$

解:

$$v(t) = L \times \frac{di(t)}{dt}$$

$$16\cos(2t) = 2 \times \tfrac{di(t)}{dt}$$ 等式兩邊同時除以 2

$$8\cos(2t) = \frac{di(t)}{dt}$$

$$8\cos(2t)dt = di(t)$$ 類似交叉相乘

$$\int_0^t 8\cos(2\tau)d\tau = \int_0^t di(\tau)$$

$$8\int_0^t \cos(2\tau)d\tau = \int_0^t di(\tau)$$

$$8\int_0^t \cos(2\tau)d(2\tau) \times \frac{1}{2} = \int_0^t di(\tau)$$

$$4\int_0^t \cos(2\tau)d(2\tau) = \int_0^t di(\tau)$$

$$4 \times (\sin(2\tau))\Big|_0^t = i(\tau)\Big|_0^t$$

$$4\sin(2t) - 0 = i(t) - i(0) = i(t) - 0$$

$$i(t) = 4\sin(2t)$$

範例 5

電機系常用的電感L其電壓v與電流i有以下關係：

$$v(t) = L \times \frac{di(t)}{dt}$$

已知$L = 4(H)$; $i(t) = -4e^{-2t}$; 求$v(t) =?$

解:

$$v(t) = L \times \frac{di(t)}{dt} = 4 \times \frac{d(-4e^{-2t})}{dt} = 4 \times (-4)\frac{de^{-2t}}{dt}$$
$$= 4 \times (-4) \times (-2)e^{-2t} = 32e^{-2t}$$

範例 6

電機系常用的電感L其電壓v與電流i有以下關係：

$$v(t) = L \times \frac{di(t)}{dt}$$

已知$L = 4(H);\ \ v(t) = 32e^{-2t}; i(0) = -4$ 求$i(t) = ?$

解:

$$v(t) = L \times \frac{di(t)}{dt}$$

$$32e^{-2t} = 4 \times \frac{di(t)}{dt}$$ 等式兩邊同時除以 4

$$8e^{-2t} = \frac{di(t)}{dt}$$

$$8\,e^{-2t}\,dt = di(t)$$ 類似交叉相乘

$$\int_0^t 8\,e^{-2\tau}\,d\tau = \int_0^t di(\tau)$$

$$8\int_0^t e^{-2\tau}\,d\tau = \int_0^t di(\tau)$$

$$8\int_0^t e^{-2\tau}\,d(-2\tau) \times \frac{1}{-2} = \int_0^t di(\tau)$$

$$-4\int_0^t e^{-2\tau}\,d(-2\tau) = \int_0^t di(\tau)$$

$$-4 \times (e^{-2\tau})\Big|_0^t = i(\tau)\Big|_0^t$$

$$-4e^{-2t} - (-4) = i(t) - i(0) = i(t) - (-4)$$

$$i(t) = -4e^{-2t}$$

範例 7

電機系常用的電感L其電壓v與電流i有以下關係：

$$v(t) = L \times \frac{di(t)}{dt}$$

已知$L = 4(H)$; $i(t) = 10$; 求$v(t) = ?$

解:

$$v(t) = L \times \frac{di(t)}{dt} = 4 \times \frac{d10}{dt} = 4 \times 0 = 0$$

<u>範例 8</u>

電機系常用的電感L其電壓v與電流i有以下關係：

$$v(t) = L \times \frac{di(t)}{dt}$$

已知$L = 4(H)$; $v(t) = 0; i(0) = 10$ 求$i(t) =?$

解:

$$v(t) = L \times \frac{di(t)}{dt}$$

$$0 = 4 \times \frac{di(t)}{dt} \text{ 等式兩邊同時除以 4}$$

$$0 = \frac{di(t)}{dt}$$

$$0 = di(t) \quad \text{類似交叉相乘}$$

$$\int_0^t 0 d\tau = \int_0^t di(\tau)$$

$$C \Big|_0^t = \int_0^t di(\tau)$$

$$C - C = i(\tau) \Big|_0^t$$

$$0 = i(t) - i(0) = i(t) - 10$$

$$i(t) = 10$$

C : Constant

$$F(s)=\mathcal{L}[f(t)]=\int_0^{\infty} f(t)e^{-st}\,dt=\int_0^{\infty} e^{-st}f(t)dt$$

<u>範例 9</u> $\int_0^{\infty} e^{-st}\times 1\,dt=\int_0^{\infty} e^{-st}dt$

$$=\int_0^{\infty} e^{-st}d(-st)\times\frac{1}{-s}$$

$$=\left.\frac{e^{-st}}{-s}\right|_0^{\infty}$$

$$=\frac{1}{s}$$

<u>範例 10</u> $\int_0^{\infty} e^{-st}\times e^{-at}\,dt=\int_0^{\infty} e^{-(s+a)t}dt$ (一般 a>0)

$$=\int_0^{\infty} e^{-(s+a)t}d(-(s+a))t\times\frac{1}{-(s+a)}$$

$$=\left.\frac{e^{-(s+a)t}}{-(s+a)}\right|_0^{\infty}$$

$$=\frac{1}{s+a}$$

範例 11

利用極座標與直角座標變換$re^{\theta i} = r\cos(\theta) + r\sin(\theta)i$

取$r = 1,\ \theta = bt$

$$1e^{bti} = 1\cos(bt) + 1\sin(bt)i \quad (\text{電機系常用} bt = \omega t)$$

$$\int_0^\infty e^{-st}\, e^{bti}\, dt = \int_0^\infty e^{-st}(\cos(bt) + \sin(bt))\, dt$$

$$= \int_0^\infty e^{-st}\cos(bt)\, dt + \int_0^\infty e^{-st}\sin(bt)dt\ i \text{ ---- (1)}$$

$$= \int_0^\infty e^{-(s-bi)t}\ dt$$

$$= \int_0^\infty e^{-(s-bi)t}\ d(-(s-bi)t) \times \frac{1}{-(s-bi)}$$

$$= \left.\frac{e^{-(s\text{-}bi)t}}{-(s-bi)}\right|_0^\infty$$

$$= -\frac{1}{-(s-bi)}$$

$$= \frac{1(s+bi)}{(s-bi)(s+bi)}$$

$$= \frac{s+bi}{s^2+b^2} = \frac{s}{s^2+b^2} + \frac{b}{s^2+b^2}\ i$$

比較(1)可得

$$\int_0^\infty e^{-st}\cos(bt)\ dt = \frac{s}{s^2+b^2}$$

$$\int_0^\infty e^{-st}\sin(bt)\ dt = \frac{b}{s^2+b^2}$$

範例 12

$$\int_0^\infty e^{-st}\, e^{-at}\, e^{bti}\, dt = \int_0^\infty e^{-st}(e^{-at}\cos(bt) + e^{-at}\sin(bt)i)dt$$

$$= \int_0^\infty e^{-(s+a-bi)t}\ dt$$

$$= \int_0^\infty e^{-st}e^{-at}\cos(bt)dt + \int_0^\infty e^{-st}e^{-at}\sin(bt)\mathrm{dt}\, i$$ --- (2)

$$= \int_0^\infty e^{-(s+a-bi)t}\ d(-(s+a-bi))t \times \frac{1}{-(s+a-bi)}$$

$$= \frac{e^{(s+a-bi)t}}{-(s+a-bi)}\Bigg|_0^\infty$$

$$= -\frac{1}{-(s+a-bi)}$$

$$= \frac{1(s+a+bi)}{(s+a-bi)(s+a+bi)}$$

$$= \frac{s+a+bi}{(s+a)^2+b^2}$$

$$= \frac{s+a}{(s+a)^2+b^2} + \frac{b}{(s+a)^2+b^2}\ i$$

比較(2)可得

$$\int_0^\infty e^{-st}e^{-at}\cos(bt)dt = \frac{s+a}{(s+a)^2+b^2}$$

$$\int_0^\infty e^{-st}e^{-at}\sin(bt)dt = \frac{b}{(s+a)^2+b^2}$$

求有效值與平均功率及能量(一個週期)會用到的積分

$$\omega = \frac{2\pi}{T}$$

<u>範例 13</u>

$\int_0^T \cos(\omega t) \times \cos(\omega t)\, dt$ (利用$\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)$)

$$= \int_0^T (\frac{1}{2} + \frac{1}{2}\cos(2\omega t))dt$$

$$= \int_0^T \frac{1}{2} dt + \int_0^T \frac{1}{2}\cos(2\omega t) dt$$

$$= \frac{1}{2}T + \frac{1}{2}\int_0^T \cos(2\omega t) d(2\omega t) \times \frac{1}{2\omega}$$

$$= \frac{1}{2}T + \frac{1}{2}\sin(2\omega t)\Big|_0^T \times \frac{1}{2\omega}$$

$$= \frac{1}{2}T$$

範例 14

$\int_0^T \sin(\omega t) \times \sin(\omega t)\, dt$　(利用$\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$)

$$= \int_0^T (\frac{1}{2} - \frac{1}{2}\cos(2\mathrm{\omega t}))\, dt$$

$$= \int_0^T \frac{1}{2}\, dt - \int_0^T \frac{1}{2}\cos(2\mathrm{\omega t})\, dt$$

$$= \frac{1}{2}T - \frac{1}{2}\int_0^T \cos(2\mathrm{\omega t})\, d(2\omega t) \times \frac{1}{2\omega}$$

$$= \frac{1}{2}T - \frac{1}{2}\sin\,(2\omega t)\Big|_0^T \times \frac{1}{2\omega}$$

$$= \frac{1}{2}T$$

範例 15　　(利用$\cos\alpha\cos\beta=\frac{1}{2}\cos(\alpha-\beta)+\frac{1}{2}\cos(\alpha+\beta)$)

$$\int_0^T \cos(\omega t)\cos(\omega t-\theta)dt$$

$$=\int_0^T (\frac{1}{2}\cos(\theta)+\frac{1}{2}\cos(2\omega t-\theta))dt$$

$$=\frac{1}{2}\int_0^T \cos\theta\ dt+\frac{1}{2}\int_0^T \cos\ (2\omega t-\theta)dt$$

$$=\frac{1}{2}\cos\theta\cdot T+\frac{1}{2}\int_0^T \cos(2\omega t-\theta)d(2\omega t)\times\frac{1}{2\omega}$$

$$=\frac{1}{2}\cos\theta\cdot T+\frac{1}{2}\int_0^T \cos(2\omega t-\theta)d(2\omega t-\theta)\times\frac{1}{2\omega}$$

$$=\frac{1}{2}\cos\theta\cdot T+\frac{1}{2}\sin(2\omega t-\theta)\Big|_0^T\times\frac{1}{2\omega}$$

$$=\frac{1}{2}\cos\theta\cdot T+\frac{1}{2}[\sin(-\theta)-\sin(-\theta)]\times\frac{1}{2\omega}$$

$$=\frac{1}{2}\cos\theta\cdot T$$

範例 16　　(利用$\sin\alpha\sin\beta=\frac{1}{2}\cos(\alpha-\beta)-\frac{1}{2}\cos(\alpha+\beta)$)

$$\int_0^T \sin(\omega t)\sin(\omega t-\theta)dt$$

$$=\int_0^T (\frac{1}{2}\cos(\theta)-\frac{1}{2}\cos(2\omega t-\theta))dt$$

$$=\frac{1}{2}\int_0^T \cos\theta\ dt-\frac{1}{2}\int_0^T \cos\ (2\omega t-\theta)dt$$

$$=\frac{1}{2}\cos\theta\cdot T-\frac{1}{2}\int_0^T \cos(2\omega t-\theta)d(2\omega t)\times\frac{1}{2\omega}$$

$$=\frac{1}{2}\cos\theta\cdot T-\frac{1}{2}\int_0^T \cos(2\omega t-\theta)d(2\omega t-\theta)\times\frac{1}{2\omega}$$

$$=\frac{1}{2}\cos\theta\cdot T-\frac{1}{2}\sin(2\omega t-\theta)\Big|_0^T\times\frac{1}{2\omega}$$

$$=\frac{1}{2}\cos\theta\cdot T-\frac{1}{2}[\sin(-\theta)-\sin(-\theta)]\times\frac{1}{2\omega}$$

$$=\frac{1}{2}\cos\theta\cdot T$$